

# On the codimension growth of simple color Lie superalgebras

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**Abstract.** We study polynomial identities of finite dimensional simple color Lie superalgebras over an algebraically closed field of characteristic zero graded by the product of two cyclic groups of order 2. We prove that the codimensions of identities grow exponentially and the rate of exponent equals the dimension of the algebra. A similar result is also obtained for graded identities and graded codimensions.

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## 1. Introduction

In this paper we begin to study numerical invariants of polynomial identities of finite dimensional simple color Lie superalgebras over an algebraically closed field of characteristic zero. Identities play an important role in the study of simple algebras. It follows from the celebrated Amitsur-Levitzky Theorem (see, for example, [8, pp.16-18]) that two finite dimensional simple associative algebras over an algebraically closed field are isomorphic if and only if they satisfy the same polynomial identities. Similar results were later obtained for Lie algebras [13], Jordan algebras [3] and some other classes. Most recent results [16] were proved for arbitrary finite dimensional simple algebras. In the associative case, finite dimensional graded simple algebras can also be uniquely defined by their graded identities [12].

An alternative approach to the characterization of finite dimensional simple algebras by their identities uses numerical invariants of identities of algebras. Given an algebra  $A$ , one can associate with it a sequence of integers  $\{c_n(A)\}$ , called codimensions of  $A$  (all definitions will be recalled in the next section). If  $\dim A = d$ , then it is well-known that  $c_n(A) \leq d^{n+1}$  (see [10]). For associative Lie and Jordan algebras it is known that  $c_n(A)$  grows asymptotically like  $t^n$ , where  $t$  is an integer and  $0 \leq t \leq d$  (see [5], [10], [15]). Moreover,  $t = d$  if and only if  $A$  is simple.

In the present paper we study the asymptotics of codimensions of color Lie superalgebras in the case when  $G = \langle a \rangle_2 \times \langle b \rangle_2 \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_2$  is the product of two

cyclic groups of order two and a skew-symmetric bicharacter  $\beta : G \times G \rightarrow F^*$  is given by  $\beta(a, a) = \beta(b, b) = 1, \beta(a, b) = -1$ . For any finite dimensional simple Lie algebra  $B$ , the corresponding color Lie superalgebra  $L = F[G] \otimes B$  is simple (see [2]). The main result of the paper asserts that the limit  $\lim_{n \rightarrow \infty} \sqrt[n]{c_n(L)}$  exists and equals  $\dim A$  (see Theorem 4.2). All necessary information about polynomial identities, codimensions and color Lie superalgebras can be found in [1], [8].

## 2. Preliminaries

Let  $F$  be a field and  $G$  a finite abelian group. An algebra  $L$  over  $F$  is said to be  $G$ -graded if

$$L = \bigoplus_{g \in G} L_g$$

where  $L_g$  is a subspace of  $L$  and  $L_g L_h \subseteq L_{gh}$ . An element  $x \in L$  is said to be homogeneous if  $x \in L_g$  for some  $g \in G$  and then we say that the degree of  $x$  in the grading is  $g$ ,  $\deg x = g$ . Any element  $x \in L$  can be uniquely decomposed into a sum  $x = x_{g_1} + \cdots + x_{g_k}$ , where  $x_{g_1} \in L_{g_1}, \dots, x_{g_k} \in L_{g_k}$  and  $g_1, \dots, g_k \in G$  are pairwise distinct. A subspace  $V \subseteq L$  is said to be homogeneous or graded subspace if for any  $x = x_{g_1} + \cdots + x_{g_k} \in V$  we have  $x_{g_1}, \dots, x_{g_k} \in V$ . A subalgebra (ideal)  $H \subseteq L$  is said to be a graded subalgebra (ideal) if it is graded as a subspace.

A map  $\beta : L \times L \rightarrow F^*$  is said to be a skew-symmetric bicharacter if

$$\beta(gh, k) = \beta(g, k)\beta(h, k), \quad \beta(g, hk) = \beta(g, h)\beta(g, k), \quad \beta(g, h)\beta(h, g) = 1.$$

A graded  $G$ -graded algebra  $L = \bigoplus_{g \in G} L_g$  is called a color Lie superalgebra or, more precisely, a  $(G, \beta)$ -color Lie superalgebra if for any homogeneous  $x, y, z \in L$  one has

$$\begin{aligned} xy &= -\beta(x, y)yx, \\ (xy)z &= x(yz) - \beta(x, y)y(xz). \end{aligned}$$

Here, for convenience, we write  $\beta(x, y)$  instead of  $\beta(\deg x, \deg y)$ . Traditionally, the product in color Lie superalgebras is written as a Lie bracket,  $xy = [x, y]$ . It is not difficult to see that  $\beta(e, g) = \beta(g, e) = 1$ , where  $e$  is the unit of  $G$  and  $\beta(g, g) = \pm 1$  for any  $g \in G$  and for any bicharacter  $\beta$ . In the case  $G = \mathbb{Z}_2$ ,  $\beta(1, 1) = -1$  we get an ordinary Lie superalgebra. If  $\beta(g, g) = 1$  for all  $g \in G$  then a  $(G, \beta)$ -color Lie superalgebra is called a color Lie algebra.

By definition, a color Lie superalgebra is simple if it has no non-trivial graded ideals. We study identical relations of  $(G, \beta)$ -color Lie algebras in the case  $G = \langle a \rangle_2 \times \langle b \rangle_2 \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_2$  and  $\beta(a, a) = \beta(b, b) = 1, \beta(a, b) = -1$ . Recently, for these  $G$  and  $\beta$  all finite dimensional simple color Lie algebras were classified under a certain weak restriction [2]. One of the series of finite dimensional simple algebras can be represented in the following way.

Let  $L = F[G] \otimes B$  be a tensor product of the group ring  $F[G]$  with the canonical  $G$ -grading, and a finite dimensional simple Lie algebra  $B$  with the trivial grading. Then  $L$  is a  $G$ -graded algebra if we set  $\deg(g \otimes x) = g$  for all  $g \in G, x \in B$ .

Given  $i, j, k, l \in \{0, 1\}$ , we define the product

$$[a^i b^j \otimes x, a^k b^l \otimes y] = (-1)^{j+k} a^{i+k} b^{j+l} \otimes [x, y] \quad (1)$$

in  $L$ . Then under the multiplication (1), an algebra  $L$  becomes a  $(G, \beta)$ -color Lie algebra. Moreover,  $L$  is a simple color Lie algebra.

**Remark 2.1.** The group algebra  $F[G]$  with the multiplication

$$(a^i b^j) * (a^k b^l) = (-1)^{j+k} a^{i+k} b^{j+l} \quad (2)$$

is isomorphic to  $M_2(F)$ , the two-by-two matrix algebra over  $F$ , if we identify  $e, a, b, ab$  with

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

respectively. □

We study non-graded identities of such an algebra  $L$ .

Next we recall the main notions of the theory of polynomial identities codimension growth (see [8]). Let  $F\{X\}$  be an absolutely free algebra over  $F$  with the countable set of free generators  $X = \{x_1, x_2, \dots\}$ . A non-associative polynomial  $f = f(x_1, \dots, x_n)$  is said to be an identity of  $F$ -algebra  $A$  if  $f(a_1, \dots, a_n) = 0$  for any  $a_1, \dots, a_n \in A$ . The set of all identities of  $A$  forms an ideal  $\text{Id}(A)$  of  $F\{X\}$  stable under all endomorphisms of  $F\{X\}$ . Denote by  $P_n = P_n(x_1, \dots, x_n)$  the subspace of  $F\{X\}$  of all multilinear polynomials in  $x_1, \dots, x_n$ . Then  $P_n \cap \text{Id}(A)$  is a subspace of all multilinear identities of  $A$  on variables  $x_1, \dots, x_n$ . A non-negative integer

$$c_n(A) = \dim \frac{P_n}{P_n \cap \text{Id}(A)}$$

is called  $n$ th codimension of  $A$ . It is well-known [10, Proposition 2] that

$$c_n(A) \leq d^{d+1} \quad (3)$$

as soon as  $\dim A = d < \infty$ . In particular, the sequence  $\sqrt[n]{c_n(A)}$  is restricted. In the 1980's, Amitsur conjectured that the limit  $\lim_{n \rightarrow \infty} \sqrt[n]{c_n(A)}$  exists and is an integer for any associative PI-algebra  $A$ . Amitsur's conjecture was confirmed for associative [6],[7], finite dimensional Lie [17] and simple special Jordan algebras [10]. For general non-associative algebras a series of counterexamples with a fractional rate of exponent were constructed in [4], [18]. If the limit exists we call it the PI-exponent of  $A$ ,

$$\text{PI-exp}(A) = \lim_{n \rightarrow \infty} \sqrt[n]{c_n(A)}.$$

### 3. Multialternating polynomials

Multialternating polynomials play an exceptional role in computing PI-exponents of simple algebras. In the associative and the Lie case one may choose multialternating polynomials among central polynomials constructed by Formanek and Razmyslov. In the Jordan case the existence of central polynomials is an open problem. Nevertheless, Razmyslov's approach (see [14]) allows one to construct the required multialternating polynomials. We shall follow the Jordan case [9], [10].

Recall that  $B$  is a finite dimensional simple Lie algebra over an algebraically closed field of characteristic zero,  $G = \langle a \rangle_2 \times \langle b \rangle_2 \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_2$  and  $\beta : G \times G \rightarrow F^*$  the skew-symmetric bicharacter on  $G$ . The simple color Lie algebra  $L$  is equal to  $F[G] \otimes B$  and the multiplication on  $L$  is defined by (1).

As in the Lie case we define the linear transformation  $\text{ad } x : L \rightarrow L$  as the right multiplication by  $x$ ,  $\text{ad } x : y \mapsto [y, x]$ . Consider the Killing form  $\rho$  on  $L$ :

$$\rho(x, y) = \text{tr}(\text{ad } x \cdot \text{ad } y).$$

**Lemma 3.1.** *The Killing form is a symmetric non-degenerate bilinear form on  $L$ .*

*Proof.* Linearity and symmetry of  $\rho$  are obvious. Fix any basis  $C = \{c_1, \dots, c_d\}$  of  $B$  where  $d = \dim c$  and consider the basis

$$\bar{C} = \{e \otimes c_i, a \otimes c_i, b \otimes c_i, ab \otimes c_i \mid 1 \leq i \leq d\}$$

of  $L$ . Let  $M$  be the matrix of  $\rho$  in this basis. Consider two basis elements  $x = g \otimes c_i, y = h \otimes c_j \in \bar{C}$  with  $g, h \in G$ . If  $g \neq h$  then  $gh \neq e$  in  $G$  and  $\text{ad } x \cdot \text{ad } y$  maps the homogeneous component  $L_t$  to  $L_{ght} \neq L_t$ . Hence  $\text{tr}(\text{ad } x \cdot \text{ad } y) = 0$ . Conversely, if  $g = h$  then any homogeneous subspace  $L_t$  is invariant under the  $\text{ad } x \cdot \text{ad } y$ -action. Moreover, if we order  $\bar{C}$  in the following way

$$\bar{C} = \{e \otimes c_1, \dots, e \otimes c_d, a \otimes c_1, \dots, a \otimes c_d, b \otimes c_1, \dots, b \otimes c_d, ab \otimes c_1, \dots, ab \otimes c_d\}$$

then  $M$  is a block-diagonal matrix with four blocks  $M_1, \dots, M_4$  on the main diagonal and all  $M_1, \dots, M_4$  are matrices of the Killing form of  $B$ . Since the Killing form on  $B$  is non-degenerate, the matrix  $M$  and  $\rho$  are also non-degenerate and we have completed the proof of the lemma.  $\square$

Now we fix our simple color Lie algebra  $L$ ,  $\dim L = q = 4 \dim B$  and construct multialternating polynomials which are not identities of  $L$ . In the rest of this section we shall assume that  $F$  is algebraically closed.

We shall use the following agreement. Given a set of indeterminates  $Y = \{y_1, \dots, y_n\}$ , we denote by  $\text{Alt}_Y$  the alternation on  $Y$ . That is, if  $f = f(x_1, \dots, x_m, y_1, \dots, y_n)$  is a polynomial multilinear on  $y_1, \dots, y_n$  then

$$\text{Alt}_Y(f) = \sum_{\sigma \in S_n} (\text{sgn } \sigma) f(x_1, \dots, x_m, y_{\sigma(1)}, \dots, y_{\sigma(n)})$$

where  $S_n$  is the symmetric group and  $\text{sgn } \sigma$  is the sign of the permutation  $\sigma \in S_n$ .

**Lemma 3.2.** *Let  $B$  be a finite dimensional simple Lie algebra,  $\dim L = d$ . Then there exists a left-normed monomial*

$$f = [x_1^1, \dots, x_{t_1}^1, y_1, x_1^2, \dots, x_{t_2}^2, y_2, \dots, x_1^d, \dots, x_{t_d}^d, y_d, x_1^{d+1}] \quad (4)$$

with  $t_1, \dots, t_d > 0$  such that  $\text{Alt}_Y(f)$  is not an identity of  $B$ .

*Proof.* By [14, Theorem 12.1] there exists a central polynomial for the pair  $(B, \text{Ad } B)$  that is an associative polynomial

$$w = w(x_1^1, \dots, x_d^1, \dots, x_1^k, \dots, x_d^k)$$

such that  $w$  is alternating on each set  $\{x_1^i, \dots, x_d^i\}$  and

$$w(\text{ad } \bar{x}_1^1, \dots, \text{ad } \bar{x}_d^k) = \lambda E$$

is a scalar linear map on  $B$  for any evaluation  $x_j^i \mapsto \bar{x}_j^i \in B$ . Moreover,  $\lambda \neq 0$  as soon as  $\bar{x}_1^i, \dots, \bar{x}_d^i$  are linearly independent for any fixed  $1 \leq i \leq d$ .

Hence  $[x_0, w]$  is not an identity of  $B$ . Here we write  $[x_0, w]$  instead of  $w(x_0) = w(\text{ad } \bar{x}_1^1, \dots, \text{ad } \bar{x}_d^k)(x_0)$ . By interrupting the alternation on all sets except  $x_1^k, \dots, x_d^k$  and renaming  $x_1^k = y_1, \dots, x_d^k = y_d$  we obtain a multilinear polynomial skew-symmetric on  $y_1, \dots, y_d$  which is not an identity of  $B$ . By rewriting this polynomial as a linear combination of left-normed monomials we can get at least one monomial of the type (4) such that  $\text{Alt}_Y(g)$  is not an identity of  $B$  but perhaps does not satisfy the condition  $t_1, \dots, t_d > 0$ .

If all  $t_1, \dots, t_d > 0$  then we are done. Suppose some  $t_i = 0$ . For brevity we assume  $t_1 = 1, t_2 = 0$ . We again use the central polynomial. Replace  $g = [x_1^1, y_1, y_2, \dots]$  with

$$g' = [x_1^1, y_1, w', y_2, \dots]$$

where  $w'$  is the central polynomial written in new variables  $\tilde{x}_j^i$  and we apply  $w'(\tilde{x}_j^i)$  to  $[x_1^1, y_1]$ . Since  $w'$  is a central polynomial, one of the left-normed monomials  $f'$  of  $g'$  is also of the form (4) with the same  $t_1, t_3, \dots, t_d$  but with  $t_2 > 0$  and  $\text{Alt}_Y(g')$  is not an identity of  $B$ . By applying this procedure at most  $d$  times we obtain a required polynomial (4). The existence of the last factor  $x_1^{d+1}$  is obvious.  $\square$

Using Lemma 3.2 we construct the first alternating polynomial for  $L = F[G] \otimes B$ .

**Lemma 3.3.** *There exists a multilinear polynomial  $f = f(x_1, \dots, x_q, y_1, \dots, y_k)$  which is not vanishing on  $L$  and is alternating on  $x_1, \dots, x_q$ .*

*Proof.* Let  $f$  be the monomial obtained in Lemma 3.2. Then there exists an evaluation  $\varphi : X \rightarrow B$ ,  $\varphi(x_j^i) = \bar{x}_j^i, \varphi(y_i) = \bar{y}_i$ , such that  $\varphi(h) \neq 0$  where  $h = \text{Alt}_Y(f)$ . Given  $1 \leq i, j \leq 2$ , we consider the evaluation  $\varphi_{ij} : X \rightarrow L$  of the following type:

$$\varphi_{ij}(y_k) = E_{ij} \otimes \bar{y}_k, \quad \varphi_{ij}(x_{t_k}^k) = E_{1i} \otimes \bar{x}_{t_k}^k, \quad \varphi_{ij}(x_1^{k+1}) = E_{j1} \otimes \bar{x}_1^{k+1}, \quad 1 \leq k \leq d,$$

and

$$\varphi_{ij}(x_s^r) = E_{11} \otimes \bar{x}_s^r$$

for all remaining  $x_s^r$  where  $E_{ij}$ 's are matrix units of  $F[G] \simeq M_2(F)$  (see Remark 2.1). Then

$$\varphi_{ij}(h) = E_{11} \otimes \varphi(h) \neq 0$$

in  $L$ . Now we write  $h$  on four disjoint sets of indeterminates,

$$h_1 = h(X_1, Y_1), \dots, h_4 = h(X_4, Y_4).$$

Since  $B$  is simple, the polynomial

$$H = [h_1, z_1^1, \dots, z_{r_1}^1, h_2, z_1^2, \dots, z_{r_2}^2, \dots, h_4]$$

is not an identity of  $L$  for some  $r_1, \dots, r_4 \geq 0$ . Moreover,

$$\varphi_0(\text{Alt}(H)) = 4d! \cdot [\varphi_{11}(h_1), \bar{z}_1^1, \dots, \bar{z}_{r_1}^1, \varphi_{12}(h_2), \dots, \varphi_{22}(h_4)] \quad (5)$$

where  $\varphi_0|_{x_1, y_1} = \varphi_{11}, \dots, \varphi_0|_{x_4, y_4} = \varphi_{22}$ ,  $\varphi_0(z_\delta^\gamma) = \bar{z}_\delta^\gamma$  and the right hand side of (5) is non-zero for some  $\bar{z}_\delta^\gamma \in L$ . Here  $\text{Alt}$  on the left hand side of (5) means the alternation on  $Y_1 \cup \dots \cup Y_4$ . Since  $|Y_1 \cup \dots \cup Y_4| = 4d = \dim L = q$ , we have completed the proof of the lemma.  $\square$

For extending the number of alternating sets of variables we shall use the following technical lemma.

**Lemma 3.4.** *Let  $f = f(x_1, \dots, x_m, y_1, \dots, y_k)$  be a multilinear polynomial alternating on  $x_1, \dots, x_m$ . Then, for  $v, z \in X$ , the polynomial*

$$g = \sum_{i=1}^m f(x_1, \dots, x_{i-1}, [x_i, v, z], x_{i+1}, \dots, x_m, y_1, \dots, y_k)$$

*is also alternating on  $x_1, \dots, x_m$ .*

*Proof.* Clearly it is enough to check that  $g$  is alternating on  $x_r, x_s$ ,  $1 \leq r < s \leq m$ . Suppose for instance that  $r = 1$  and  $s = 2$ . Since the polynomial

$$\sum_{i=3}^m f(x_1, \dots, [x_i, v, z], \dots, x_m, y_1, \dots, y_k)$$

is alternating on  $x_1$  and  $x_2$ , it is enough to check that

$$g' = f([x_1, v, z], x_2, \dots, x_m, y_1, \dots, y_k) + f(x_1, [x_2, v, z], x_3, \dots, x_m, y_1, \dots, y_k)$$

is alternating on  $x_1$  and  $x_2$ . But

$$\begin{aligned} g'(x_1, x_2, \dots) + g'(x_2, x_1, \dots) &= f([x_1, v, z], x_2, \dots) \\ &+ f(x_1, [x_2, v, z], \dots) + f([x_2, v, z], x_1, \dots) + f(x_2, [x_1, v, z], \dots) \end{aligned}$$

$$= f([x_1, v, z], x_2, \dots) - f([x_2, v, z], x_1, \dots) + \\ f([x_2, v, z], x_1, \dots) - f([x_1, v, z], x_2, \dots) \equiv 0,$$

since  $f(x, y, \dots) = -f(y, x, \dots)$ . □

In order to simplify the notation, we shall often write  $f = f(x_1, \dots, x_m, y_1, \dots, y_n) = f(x_1, \dots, x_m, Y)$ , where  $Y = \{y_1, \dots, y_n\}$ .

**Lemma 3.5.** *Let  $Y = Y_0 \cup Y_1 \cup \dots \cup Y_r \subseteq X$  be a disjoint union with  $r \geq 0$  and  $Y_0$  eventually empty. Let  $f = f(x_1, \dots, x_q, Y)$  be a multilinear polynomial alternating on each  $Y_i$ ,  $1 \leq i \leq r$ , and on  $x_1, \dots, x_q$ . Then, for any  $k \geq 1$  and for any  $v_1, z_1, \dots, v_k, z_k \in X$ , there exists a multilinear polynomial*

$$g = g(x_1, \dots, x_q, v_1, z_1, \dots, v_k, z_k, Y)$$

such that, for any evaluation  $\varphi : X \rightarrow L$ ,  $\varphi(x_i) = \bar{x}_i$ ,  $1 \leq i \leq q$ ,  $\varphi(v_j) = \bar{v}_j$ ,  $\varphi(z_j) = \bar{z}_j$ ,  $1 \leq j \leq k$ ,  $\varphi(y) = \bar{y}$ , for  $y \in Y$ , we have

$$\varphi(g) = g(\bar{x}_1, \dots, \bar{x}_q, \bar{v}_1, \bar{z}_1, \dots, \bar{v}_k, \bar{z}_k, \bar{Y})$$

$$= \text{tr}(\text{ad } v_1 \cdot \text{ad } z_1) \cdots \text{tr}(\text{ad } v_k \cdot \text{ad } z_k) f(\bar{x}_1, \dots, \bar{x}_q, \bar{Y}).$$

Moreover  $g$  is alternating on each set  $Y_i$ ,  $1 \leq i \leq r$ , and on  $x_1, \dots, x_q$ .

*Proof.* The proof is by induction of  $k$ . Suppose first that  $k = 1$  and define

$$g = g(x_1, \dots, x_q, v, z, Y) = \sum_{i=1}^q f(x_1, \dots, [x_i, v, z], \dots, x_q, Y).$$

Then  $g$  is alternating on each set  $Y_i$ ,  $1 \leq i \leq r$  and, by Lemma 3.4, is also alternating on  $x_1, \dots, x_q$ . Consider an evaluation  $\varphi : X \rightarrow L$  such that  $\varphi(x_i) = \bar{x}_i$ ,  $1 \leq i \leq q$ ,  $\varphi(v) = \bar{v}$ ,  $\varphi(z) = \bar{z}$ ,  $\varphi(y) = \bar{y}$ , for  $y \in Y$ . Suppose first that the elements  $\bar{x}_1, \dots, \bar{x}_q$  are linearly dependent over  $F$ . Then, since  $f$  and  $g$  are alternating on  $x_1, \dots, x_q$ , it follows that  $\varphi(f) = \varphi(g) = 0$  and we are done.

Therefore we may assume that  $\bar{x}_1, \dots, \bar{x}_q$  are linearly independent over  $F$  and so, since  $\dim L = q$ , they form a basis of  $L$ . Hence for all  $i = 1, \dots, q$ , we write

$$[\bar{x}_i \bar{v}, \bar{z}] = \alpha_{ii} \bar{x}_i + \sum_{j \neq i} \alpha_{ij} \bar{x}_j,$$

for some scalars  $\alpha_{ij} \in F$ . Since  $f$  is alternating on  $x_1, \dots, x_q$ ,

$$f(\bar{x}_1, \dots, [\bar{x}_i, \bar{v}, \bar{z}], \dots, \bar{x}_q, \bar{Y}) = \alpha_{ii} f(\bar{x}_1, \dots, \bar{x}_i, \dots, \bar{x}_q, \bar{Y}).$$

Therefore

$$g(\bar{x}_1, \dots, \bar{x}_q, \bar{v}, \bar{z}, \bar{Y}) = (\alpha_{11} + \dots + \alpha_{qq}) f(\bar{x}_1, \dots, \bar{x}_q, \bar{Y}),$$

and, since  $\alpha_{11} + \dots + \alpha_{qq} = \text{tr}(\text{ad } v \cdot \text{ad } z)$ , the lemma is thus proved in case  $k = 1$ .

Now let  $k > 1$  and let  $g = g(x_1, \dots, x_q, v_1, z_1, \dots, v_{k-1}, z_{k-1}, Y)$  be a multilinear polynomial satisfying the conclusion of the lemma. Then we write  $g =$

$g(x_1, \dots, x_q, Y')$ , where  $Y' = Y'_0 \cup Y_1 \cup \dots \cup Y_r$  and  $Y'_0 = Y_0 \cup \{v_1, z_1, \dots, v_{k-1}, z_{k-1}\}$ . If we now apply to  $g$  the same arguments as in the case  $k = 1$ , we obtain a polynomial satisfying the conclusion of the lemma.  $\square$

Now we are ready to construct the required multialternating polynomial for our simple color Lie algebra  $L$ ,  $\dim L = q$ . Recall that  $F$  is an algebraically closed field of characteristic zero.

**Proposition 3.6.** *For any  $k \geq 0$  there exists a multilinear polynomial*

$$g_k = g_k(x_1^{(1)}, \dots, x_q^{(1)}, \dots, x_1^{(2k+1)}, \dots, x_q^{(2k+1)}, y_1, \dots, y_N)$$

satisfying the following conditions:

- 1)  $g_k$  is alternating on each set  $\{x_1^{(i)}, \dots, x_q^{(i)}\}$ ,  $1 \leq i \leq 2k+1$ ;
- 2)  $g_k$  is not the identity of  $L$ ;
- 3) the integer  $N$  does not depend on  $k$ .

*Proof.* Let  $f = f(x_1, \dots, x_q, y_1, \dots, y_m)$  be the multilinear polynomial from Lemma 3.3. Hence  $f$  is alternating on  $x_1, \dots, x_q$  and does not vanish on  $L$ .

Suppose first that  $k = 1$  and write  $Y = \{y_1, \dots, y_m\}$ . By Lemma 3.5 there exists a multilinear polynomial

$$g = g(x_1, \dots, x_q, v_1^{(1)}, z_1^{(1)}, \dots, v_q^{(1)}, z_q^{(1)}, Y)$$

such that

$$\begin{aligned} & g(\bar{x}_1, \dots, \bar{x}_q, \bar{v}_1^{(1)}, \bar{z}_1^{(1)}, \dots, \bar{v}_q^{(1)}, \bar{z}_q^{(1)}, \bar{Y}) \\ &= \text{tr}(\text{ad } \bar{v}_1^{(1)} \cdot \text{ad } \bar{z}_1^{(1)}) \cdots \text{tr}(\text{ad } \bar{v}_q^{(1)} \cdot \text{ad } \bar{z}_q^{(1)}) f(\bar{x}_1, \dots, \bar{x}_q, \bar{Y}). \end{aligned}$$

Now, for any  $\sigma, \tau \in S_q$ , define the polynomial

$$\begin{aligned} g_{\sigma, \tau} &= g_{\sigma, \tau}(x_1, \dots, x_q, v_1^{(1)}, z_1^{(1)}, \dots, v_q^{(1)}, z_q^{(1)}, Y) \\ &= g(x_1, \dots, x_q, v_{\sigma(1)}^{(1)}, z_{\tau(1)}^{(1)}, \dots, v_{\sigma(q)}^{(1)}, z_{\tau(q)}^{(1)}, Y). \end{aligned}$$

Then set

$$g_1(x_1, \dots, x_q, v_1^{(1)}, z_1^{(1)}, \dots, v_q^{(1)}, z_q^{(1)}, Y) = \frac{1}{q!} \sum_{\sigma, \tau \in S_q} (\text{sgn } \sigma)(\text{sgn } \tau) g_{\sigma, \tau}.$$

The polynomial  $g_1$  is alternating on each of the sets  $\{x_1, \dots, x_q\}$ ,  $\{v_1^{(1)}, \dots, v_q^{(1)}\}$  and  $\{z_1^{(1)}, \dots, z_q^{(1)}\}$ . Next we show that for any evaluation  $\varphi$ ,

$$\varphi(g_1) = \det \bar{\rho}_1 \cdot \varphi(f),$$



where

$$\bar{\rho}_1 = \begin{pmatrix} \rho(\bar{v}_1^{(1)}, \bar{z}_1^{(1)}) & \cdots & \rho(\bar{v}_1^{(1)}, \bar{z}_q^{(1)}) \\ \vdots & & \vdots \\ \rho(\bar{v}_q^{(1)}, \bar{z}_1^{(1)}) & \cdots & \rho(\bar{v}_q^{(1)}, \bar{z}_q^{(1)}) \end{pmatrix}.$$

By Lemma 3.5,

$$\varphi(g_1) = \gamma \varphi(f)$$

for any evaluation  $\varphi : X \rightarrow L$ , where

$$\gamma = \frac{1}{q!} \sum_{\sigma, \tau \in S_q} (\text{sgn } \sigma)(\text{sgn } \tau) \rho(\bar{v}_{\sigma(1)}^{(1)}, \bar{z}_{\tau(1)}^{(1)}) \cdots \rho(\bar{v}_{\sigma(q)}^{(1)}, \bar{z}_{\tau(q)}^{(1)}).$$

We fix  $\sigma \in S_q$  and we compute the sum

$$\gamma_\sigma = \sum_{\tau \in S_q} (\text{sgn } \tau) \rho(\bar{v}_{\sigma(1)}^{(1)}, \bar{z}_{\tau(1)}^{(1)}) \cdots \rho(\bar{v}_{\sigma(q)}^{(1)}, \bar{z}_{\tau(q)}^{(1)}).$$

Write simply  $\bar{v}_{\sigma(i)}^{(1)} = a_i$ ,  $\bar{z}_i^{(1)} = b_i$ ,  $i = 1, \dots, q$ . Then

$$\begin{aligned} \gamma_\sigma &= \sum_{\tau \in S_q} (\text{sgn } \tau) \rho(a_1, b_{\tau(1)}) \cdots \rho(a_q, b_{\tau(q)}) = \det \begin{pmatrix} \rho(a_1, b_1) & \cdots & \rho(a_1, b_q) \\ \vdots & & \vdots \\ \rho(a_q, b_1) & \cdots & \rho(a_q, b_q) \end{pmatrix} \\ &= (\text{sgn } \sigma) \det \begin{pmatrix} \rho(a_{\sigma^{-1}(1)}, b_1) & \cdots & \rho(a_{\sigma^{-1}(1)}, b_q) \\ \vdots & & \vdots \\ \rho(a_{\sigma^{-1}(q)}, b_1) & \cdots & \rho(a_{\sigma^{-1}(q)}, b_q) \end{pmatrix} = (\text{sgn } \sigma) \det \bar{\rho}_1. \end{aligned}$$

Hence

$$\gamma = \frac{1}{q!} \sum_{\sigma \in S_q} (\text{sgn } \sigma) \gamma_\sigma = \det \bar{\rho}_1$$

and  $\varphi(g_1) = \det \bar{\rho}_1 \cdot \varphi(f)$ . Thus, since  $\rho$  is a non degenerate form,  $g_1$  does not vanish in  $L$ . This completes the proof in case  $k = 1$ .

If  $k > 1$ , by the inductive hypothesis there exists a multilinear polynomial

$$g_{k-1}(x_1, \dots, x_q, v_1^{(1)}, z_1^{(1)}, \dots, v_q^{(1)}, z_q^{(1)}, \dots, v_1^{(k-1)}, z_1^{(k-1)}, \dots, v_q^{(k-1)}, z_q^{(k-1)}, Y)$$

satisfying the conclusion of the theorem. We now write

$$g_{k-1} = g_{k-1}(x_1, \dots, x_q, Y'),$$

where  $Y' = Y \cup \{v_1^{(1)}, z_1^{(1)}, \dots, v_q^{(1)}, z_q^{(1)}, \dots, v_1^{(k-1)}, z_1^{(k-1)}, \dots, v_q^{(k-1)}, z_q^{(k-1)}\}$  and we apply Lemma 3.5 and the previous arguments to  $g_{k-1}$ . In this way we can construct the polynomial  $g_k$  and, for any evaluation  $\varphi$ , we have

$$\varphi(g_k) = \det \bar{\rho}_k \cdot \varphi(g_{k-1}) = \det \bar{\rho}_1 \cdots \det \bar{\rho}_k \cdot \varphi(f),$$

where

$$\bar{\rho}_s = \begin{pmatrix} \rho(\bar{v}_1^{(s)}, \bar{z}_1^{(s)}) & \cdots & \rho(\bar{v}_1^{(s)}, \bar{z}_q^{(s)}) \\ \vdots & & \vdots \\ \rho(\bar{v}_q^{(s)}, \bar{z}_1^{(s)}) & \cdots & \rho(\bar{v}_q^{(s)}, \bar{z}_q^{(s)}) \end{pmatrix},$$

for all  $1 \leq s \leq k$ . This completes the proof of the proposition.  $\square$

#### 4. PI-exponents of simple color Lie algebras

For computing PI-exponents of simple color Lie algebras we need to get a reasonable lower bound of codimension growth.

**Proposition 4.1.** *Let  $L$  be as in the previous section. Then for all  $n \geq 1$ , there exist constants  $C > 0$  and  $t$  such that*

$$Cn^t q^n \leq c_n(L), \quad (6)$$

where  $q = \dim L$ .

*Proof.* The main tool for proving the inequality (6) is the representation theory of symmetric groups. We refer reader to [11] for details of this theory.

Recall that  $P_m$  is a subspace of  $F\{X\}$  consisting of all multilinear polynomials in  $x_1, \dots, x_m$  and  $\text{Id}(L)$  is the ideal of all multilinear identities of  $L$  of degree  $m$ . One can define the  $S_m$ -action on  $P_m$  by setting

$$\sigma f(x_1, \dots, x, m) = f(x_{\sigma(1)}, \dots, x_{\sigma(m)}).$$

Then  $P_m$  becomes an  $F[S_m]$ -module and  $P_m(L) = P_m / P_m \cap \text{Id}(L)$  is its submodule. By Mashke's Theorem  $P_m(L)$  is the direct sum of irreducible components and for proving the inequality (6) it is sufficient to find at least one irreducible component with the dimension greater or equal to  $Cn^t q^n$ . Slightly modifying this approach we first consider  $P_{n+N}(L)$  where  $n = (2k+1)q + N$  and  $k, N$  are as in Proposition 3.6.

Recall that there exists 1-1 correspondence between isomorphism classes of irreducible  $S_n$ -representations and partitions of  $n$  (or Young diagrams with  $n$  boxes). A partition  $\lambda \vdash n$  is an ordered set of integers  $\lambda = (\lambda_1, \dots, \lambda_t)$  satisfying  $\lambda_1 \geq \dots \geq \lambda_t > 0$  and  $\lambda_1 + \dots + \lambda_t = n$ . The corresponding Young diagram  $D_\lambda$  is a tableau with  $n$  boxes. The first row of  $D_\lambda$  contains  $\lambda_1$  boxes, the second row contains  $\lambda_2$  boxes, and so on. Young tableau  $T_\lambda$  is the diagram  $D_\lambda$  filled up by integers  $1, \dots, n$ .

Given a Young tableau  $T_\lambda$  of shape  $\lambda \vdash n$ , let  $R_{T_\lambda}$  and  $C_{T_\lambda}$  denote the subgroups of  $S_n$  stabilizing the rows and the columns of  $T_\lambda$ , respectively. If we set

$$\bar{R}_{T_\lambda} = \sum_{\sigma \in R_{T_\lambda}} \sigma \quad \text{and} \quad \bar{C}_{T_\lambda} = \sum_{\tau \in C_{T_\lambda}} (\text{sgn} \tau) \tau.$$

then the element  $e_{T_\lambda} = \bar{R}_{T_\lambda} \bar{C}_{T_\lambda}$  is an essential idempotent of the group algebra  $FS_n$  (i.e.  $e_{T_\lambda}^2 = \gamma e_{T_\lambda}$  for some  $0 \neq \gamma \in F$ ) and  $F[S_n]e_{T_\lambda}$  is an irreducible left  $F[S_n]$ -module associated to  $\lambda$ .

By Proposition 3.6, for any fixed  $k \geq 1$  there exists a multilinear polynomial

$$g_k = g_k(x_1^{(1)}, \dots, x_q^{(1)}, \dots, x_1^{(2k+1)}, \dots, x_q^{(2k+1)}, y_1, \dots, y_N)$$

such that  $g_k$  is alternating on each set of indeterminates  $\{x_1^{(i)}, \dots, x_q^{(i)}\}$ ,  $1 \leq i \leq 2k+1$ , and  $g_k$  is not a polynomial identity of  $L$ . Rename the variables and write

$$g_k = h(x_1, \dots, x_{q(2k+1)}, Y),$$

where  $Y = \{y_1, \dots, y_N\}$ .

Since  $h \notin \text{Id}(L)$ , there exists a partition  $\lambda = (\lambda_1, \dots, \lambda_m) \vdash n$  and a Young tableau  $T_\lambda$  such that  $F[S_n]e_{T_\lambda}h \not\subseteq \text{Id}(L)$ . Our next goal is to show that  $\lambda = ((2k+1)^q)$  is a rectangle of width  $2k+1$  and height  $q$ .

If  $\lambda_1 \geq 2k+2$ , then  $e_{T_\lambda}h$  is a polynomial symmetric in at least  $2k+2$  variables among  $x_1, \dots, x_n$ . But for any  $\sigma \in \bar{R}_{T_\lambda}$  these variables in  $\sigma\bar{C}_{T_\lambda}$  are divided into  $2k+1$  disjoint alternating subsets. It follows that  $\sigma\bar{C}_{T_\lambda}h$  is alternating and symmetric in at least two variables and so,  $e_{T_\lambda}h = 0$  is the zero polynomial, a contradiction. Thus  $\lambda_1 \leq 2k+1$ .

Suppose now that  $m \geq q+1$ . Since the first column of  $T_\lambda$  is of height at least  $q+1$ , the polynomial  $\bar{C}_{T_\lambda}h$  is alternating in at least  $q+1$  variables among  $x_1, \dots, x_n$ . Since  $\dim L = q$  we get that for any  $\sigma$ ,  $\sigma\bar{C}_{T_\lambda}h \equiv 0$  on  $L$  and so, also  $e_{T_\lambda}h = \bar{R}_{T_\lambda}\bar{C}_{T_\lambda}h \equiv 0$  on  $L$ , a contradiction.

We have proved that  $F[S_n]e_{T_\lambda}h \not\subseteq \text{Id}(L)$ , for some Young tableau  $T_\lambda$  of shape  $\lambda = ((2k+1)^q)$ . From the Hook formula for dimensions of irreducible representations of  $S_n$  (see [11]) and Stirling formula for factorials it follows that

$$\dim F[S_n]e_{T_\lambda}h \geq \frac{q!}{(2\pi n)^q} q^n.$$

It easily follows from the simplicity of tensor factor  $B$  in  $L = F[G] \otimes B$  that

$$c_{n'}(L) \geq c_n(L) \quad \text{as soon as} \quad n' > n. \quad (7)$$

Hence

$$c_m(L) \geq \frac{C'}{(m-N)^q} q^{m-N} \geq \frac{C''}{m^q} q^m$$

for some constants  $C', C''$  for any  $m = q(2k+1) + N$ ,  $k = 1, 2, \dots$ . Finally, applying again the inequality (7) we get (6) for all  $n$ .  $\square$

Combining the inequality (3) and Proposition 4.1 we immediately obtain the main result of the paper.

**Theorem 4.2.** *Let  $F$  be an algebraically closed field of characteristic zero and let  $L = F[G] \otimes B$  be a finite dimensional color Lie superalgebra over  $F$  where  $G = \langle a \rangle_2 \times \langle b \rangle_2 \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_2$  with the skew-symmetric bicharacter  $\beta$  defined by  $\beta(a) = \beta(b) = 1, \beta(a, b) = -1$  and  $B$  is a finite dimensional simple Lie algebra with the trivial  $G$ -grading. Then the PI-exponent of  $L$  exists and  $\exp(L) = \dim L$ .*

## 5. Graded identities of simple color Lie algebras

In conclusion we discuss codimensions behavior of algebras defined distinct bicharacters and asymptotics of graded codimensions. We begin by an easy remark.

**Remark 5.1.** If  $L = F[G] \otimes B$  and  $G \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_2$  with the trivial bicharacter  $\beta$ , that is  $\beta \equiv 1$ , then  $\text{PI-exp}(L) = d = \frac{1}{4} \dim L$ , where  $d = \dim B$ .

*Proof.* Since  $F[G]$  is a commutative ring in this case,  $L$  is an ordinary Lie algebra with the same identities as  $B$ . In particular,  $\text{PI-exp}(L) = \text{PI-exp}(B) = \dim B$  (see [5] or [17]).  $\square$

Remark 5.1 shows that ordinary codimensions behavior strongly depends on bicharacter  $\beta$ . On the other hand one can consider graded identities of  $L$  since  $L$  is a  $G$ -graded algebra.

Recall that if we define  $G$ -grading on an infinite generating set  $Y$ , i.e. split  $Y$  to a disjoint union  $Y = \cup_{g \in G} Y^g$ ,  $\deg y^g = g \quad \forall y^g \in Y^g$ , then  $F\{Y\}$  can be endowed by the induced grading if we set  $\deg(y_{i_1} \cdots y_{i_m}) = \deg y_{i_1} \cdots \deg y_{i_m}$  for any arrangement of brackets. The polynomial  $f(y_1^{g_1}, \dots, y_n^{g_n})$  is called a graded identity of  $L$  if  $f(u_1, \dots, u_n) = 0$ , as soon as  $\deg u_1 = \deg y_1^{g_1} = g_1, \dots, \deg u_n = \deg y_n^{g_n} = g_n$ .

Since  $F\{Y\}$  is a graded algebra, the subspace of multilinear polynomials  $P_n$  should be replaced by a graded subspace

$$\bigoplus_{k_1 + \dots + k_4 = n} P_{k_1, k_2, k_3, k_4}$$

where  $P_{k_1, k_2, k_3, k_4}$  is a subspace of multilinear polynomials  $f$  on

$$y_1^{g_1}, \dots, y_{k_1}^{g_1}, \dots, y_1^{g_{k_4}}, \dots, y_{k_4}^{g_{k_4}}.$$

Graded codimensions are defined as

$$c_n^{gr}(L) = \sum_{\substack{k_1 \geq 0, k_2 \geq 0, k_3 \geq 0, k_4 \geq 0 \\ k_1 + k_2 + k_3 + k_4 = n}} \binom{n}{k_1, k_2, k_3, k_4} \dim \frac{P_{k_1, k_2, k_3, k_4}}{P_{k_1, k_2, k_3, k_4} \cap \text{Id}(L)}$$

(see [8] for details). For our class of algebras graded codimensions behavior does not depend on bicharacter  $\beta$  defining color on  $L = F[G] \otimes B$ . In the proof of the next result we shall use the following easy observation.

**Remark 5.2.** If  $\text{Lie}(X)$  is a free Lie algebra on the countable set of generators and  $B$  is an arbitrary Lie algebras then

$$\frac{P_n}{P_n \cap \text{Id}(B)} = \frac{V_n}{V_n \cap \text{Id}^{\text{Lie}}(B)}$$

where  $V_n$  is a subspace of  $\text{Lie}(X)$  of all multilinear polynomials in variables  $x_1, \dots, x_n$  and  $\text{Id}^{\text{Lie}}(B)$  is the ideal of Lie identities of  $B$  in  $\text{Lie}(X)$ .

$\square$

We need this remark since all previous results concerning codimension growth of Lie algebras were proved for Lie codimensions.

**Theorem 5.3.** Let  $F$  be an algebraically closed field of characteristic zero and let  $L = F[G] \otimes B$  be a finite dimensional color Lie superalgebra over  $F$  where

$G = \langle a \rangle_2 \times \langle b \rangle_2 \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_2$  with any skew-symmetric bicharacter  $\beta$  and  $B$  is a finite dimensional simple Lie algebra with the trivial  $G$ -grading. Then graded PI-exponent of  $L$

$$PI\text{-}exp^{gr}(L) = \lim_{n \rightarrow \infty} \sqrt[n]{c_n^{gr}(L)}$$

exists and is equal to  $|G| \dim L = 4 \dim L$ .

*Proof.* First we prove that

$$\dim \frac{P_{k_1, k_2, k_3, k_4}}{P_{k_1, k_2, k_3, k_4} \cap \text{Id}(L)} = c_n(B). \quad (8)$$

Note that any multilinear Lie polynomial in  $x_1, \dots, x_n$  can be written as a linear combination of left-normed monomials

$$m_\sigma = [x_1, x_{\sigma(2)}, \dots, x_{\sigma(n)}],$$

where  $\sigma$  is a permutation of  $2, \dots, n$ . If  $w_1, \dots, w_n$  are homogeneous elements of the color Lie superalgebra  $L = F[G] \otimes B$  then any multilinear polynomial expression on  $w_1, \dots, w_n$  is also a linear combination of left-normed products  $[w_1, w_{\sigma(2)}, \dots, w_{\sigma(n)}]$ . Since we are interested in graded identities of  $L$  it is sufficient to consider only left-normed monomials and their linear combinations.

Let  $(g_1, \dots, g_n)$  be a  $n$ -tuple of elements of  $G$  such that

$$(g_1, \dots, g_n) = (\underbrace{e, \dots, e}_{k_1}, \underbrace{a, \dots, a}_{k_2}, \underbrace{b, \dots, b}_{k_3}, \underbrace{ab, \dots, ab}_{k_4}).$$

Then

$$m_\sigma(g_1 \otimes x_1, \dots, g_n \otimes x_n) = g_1 \cdots g_n \otimes \lambda_\sigma m_\sigma(x_1, \dots, x_n)$$

in  $F[G] \otimes F\{X\}$  where  $\lambda_\sigma = \pm 1$  depends only on  $\sigma$  for given  $g_1, \dots, g_n$ .

Given a multilinear polynomial

$$f = f(x_1, \dots, x_n) = \sum_{\sigma \in S_{n-1}} \alpha_\sigma m_\sigma$$

of  $F\{X\}$  we denote by  $\tilde{f}$  the element

$$\tilde{f} = \tilde{f}(x_1, \dots, x_n) = \sum_{\sigma \in S_{n-1}} \lambda_\sigma \alpha_\sigma m_\sigma.$$

Then for any  $w_1^1, \dots, w_{k_1}^1, \dots, w_1^4, \dots, w_{k_4}^4 \in B$  we have

$$\begin{aligned} & f(e \otimes w_1^1, \dots, e \otimes w_{k_1}^1, \dots, ab \otimes w_1^4, \dots, ab \otimes w_{k_4}^4) \\ &= a^{k_2} b^{k_3} (ab)^{k_4} \otimes \tilde{f}(w_1^1, \dots, w_{k_1}^1, \dots, w_1^4, \dots, w_{k_4}^4). \end{aligned}$$

In particular,  $f$  is a graded identity of  $L$ ,  $f \in P_{k_1, k_2, k_3, k_4} \cap \text{Id}(L)$  if and only if  $f$  is an identity of the Lie algebra  $B$ . Now, if  $c_n(B) = N$  and  $m_{\sigma_1}, \dots, m_{\sigma_N}$

is a basis of  $V_n$  in  $Lie(X)$  modulo  $Id^{Lie}(B)$  then also  $m_{\sigma_1}, \dots, m_{\sigma_N}$  is a basis of  $P_{k_1, \dots, k_4}$  modulo  $Id(L)$  in  $F\{X\}$  and we have proved the relation (8). Hence

$$\begin{aligned} c_n^{gr}(L) &= \sum_{\substack{k_1 \geq 0, k_2 \geq 0, k_3 \geq 0, k_4 \geq 0 \\ k_1 + k_2 + k_3 + k_4 = n}} \binom{n}{k_1, k_2, k_3, k_4} \dim \frac{P_{k_1, k_2, k_3, k_4}}{P_{k_1, k_2, k_3, k_4} \cap Id(L)} \\ &= c_n(B) \sum_{\substack{k_1 \geq 0, k_2 \geq 0, k_3 \geq 0, k_4 \geq 0 \\ k_1 + k_2 + k_3 + k_4 = n}} \binom{n}{k_1, k_2, k_3, k_4} = 4^n c_n(B) \end{aligned}$$

and we have completed the proof since  $\lim_{n \rightarrow \infty} \sqrt[n]{c_n(B)} = \dim B$  by [5].  $\square$

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